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Drag on a sphere moving slowly in a rotating viscous fluid

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The mobility of a sphere moving slowly along the axis of a rotating, viscous, imcompressible fluid has been calculated for zero Reynolds number R and values of the Taylor number T ranging from zero to infinity, using a method of induced forces. For small values of T the mobility has been expanded in a power series in $T^{\frac{1}{2}}$; the first seven terms of this series have been evaluated. Very good agreement is found with experimental data, which are only available for $T \leqslant 0.75$.

1. Introduction

The drag force experienced by a sphere that moves along the axis of a rotating incompressible viscous fluid depends in a quite complicated way on the velocity U of the sphere and the angular velocity Ω of the fluid far away from the sphere. This conclusion may be drawn from experiments carried out by Maxworthy (1965, 1970). The present-day theoretical insight in this phenomenon is rather limited, since the fluid equations are solvable only after drastic simplification.

Among the first to consider motion in a rotating fluid were Proudman (1916), Taylor (1922) and Grace (1926). The latter obtained a formula for the ultimate drag on a sphere, started impulsively in an inviscid incompressible fluid, albeit with an estimated numerical coefficient. Many years later, Stewartson (1952) derived the exact expression.

Morrison & Morgan (1956) and Moore & Saffman (1969), among others, included viscosity, but considered steady motion in a rapidly rotating fluid. Their result for the drag on a sphere is identical with Stewartson's. Childress (1964) studied the motion of a sphere in a viscous fluid in a different regime, in which both the Taylor number $T \equiv \rho\Omega a^2/\eta$ and the Reynolds number $R \equiv \rho U a/\eta$ are small (a is the radius of the sphere; ρ and η are the density and viscosity of the fluid). He was able to determine a first correction to the Stokes drag, proportional to $T^{\frac{1}{2}}$. Recently Dennis, Ingham & Singh (1982) solved the fluid equations of motion numerically, and calculated the drag for $T \leqslant 0.5$ and $R \leqslant 0.5$. In all treatments mentioned above explicit solutions were constructed for the velocity field and pressure field in the entire fluid. Subsequently the drag was calculated by integration of the normal component of the pressure tensor over the surface of the sphere.

In this paper we shall evaluate the friction assuming that all momentum convection in the fluid may be neglected, i.e. for zero Reynolds number. We do not impose any restriction on the value of the Taylor number, so that the results also cover a range not considered before. Our analysis makes use of a method of induced forces, which was developed by Mazur & van Saarloos (1982) to analyse many-sphere hydrodynamic interactions in Stokes flow, and was applied by Mazur and the present author (Mazur

& Weisenborn 1984; Weisenborn & Mazur 1984) to evaluate the Oseen drag on a circular cylinder and a sphere. This method evades the need of constructing explicit solutions for the fluid fields.

In §2 we briefly discuss the equations of motion and continuity for the fluid.

In §3 we introduce an induced force density on the sphere in the equation of motion and give the formal solution for the velocity field in wavevector representation, in terms of this induced force density.

In §4 we expand the induced force density in irreducible force multipoles. Applying the boundary condition to the so-called velocity surface moments, we derive a set of coupled linear equations for the force multipoles. We then use this set to obtain a formal expression for the translational mobility, the inverse of the friction. With this expression we may evaluate the transitional mobility on the basis of all multipoles up to an arbitrary order, while neglecting the contributions of all multipoles of higher order. The actual evaluation, performed on the basis of the first five multipoles, shows that the multipole expansion converges rapidly: values for the mobility, based on the first three force-multipoles, are for all Taylor numbers altered by less than 1% if the influence of the fifth force-multipole is also accounted for. We further derive an alternative expression which enables us to calculate the first seven terms in the expansion of the translational mobility in powers of $T^{\frac{1}{2}}$. Finally we give the corresponding expression for the rotational mobility for the case where the applied torque is parallel to the rotation axis of the fluid, and we calculate with this expression the first three terms of the expansion of the rotational mobility in powers of $T^{\frac{1}{2}}$.

A discussion of the results is given in §5. This discussion includes a comparison with theoretical results obtained previously as well as to experimental data where available.

2. The equations of motion and continuity

We consider a sphere of radius a that moves with constant velocity $\boldsymbol{U}$ along the axis of a rotating fluid. If unperturbed by the presence of the sphere, this viscous, imcompressible and unbounded fluid rotates uniformly with constant angular velocity $\boldsymbol{\Omega}$. In a non-rotating frame of reference the equation of motion (the Navier–Stokes equation) and the equation of continuity read

$$\left.\begin{aligned} \rho\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(\boldsymbol{r},t) &= \boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t), && (2.1)\\ \boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{r},t) &= 0, && (2.2)\end{aligned}\right\} \quad \text{for } |\boldsymbol{r}-\boldsymbol{R}(t)|>a,$$

with

$$P_{\alpha\beta} = p\delta_{\alpha\beta}-\eta\left(\frac{\partial v_\alpha}{\partial r_\beta}+\frac{\partial v_\beta}{\partial r_\alpha}\right). \tag{2.3}$$

Here $\mathrm{d}/\mathrm{d}t$ is the substantial time derivative, $\boldsymbol{v}$ the velocity field, $\boldsymbol{P}$ the pressure tensor, p the hydrostatic pressure, and η and ρ the viscosity and density of the fluid. $\boldsymbol{R}(t)$ denotes the position of the centre of the sphere.

Upon transformation to a frame of reference that corotates with the unperturbed fluid, (2.1), combined with (2.3), becomes

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(\boldsymbol{r},t)+2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r},t) = -\boldsymbol{\nabla}p^*(\boldsymbol{r},t)+\eta\,\Delta\boldsymbol{v}(\boldsymbol{r},t) \quad \text{for } |\boldsymbol{r}-\boldsymbol{R}(t)|>a. \tag{2.4}$$

The reduced hydrostatic pressure $p^*(\boldsymbol{r},t)$ is defined as

$$p^*(\boldsymbol{r},t) \equiv p(\boldsymbol{r},t) - \tfrac{1}{2}\rho(\Omega^2 r^2 - (\boldsymbol{\Omega}\cdot\boldsymbol{r})^2). \tag{2.5}$$

In (2.4) and (2.5), $\mathrm{d}/\mathrm{d}t$, $\boldsymbol{v}$ and $\boldsymbol{r}$ denote the substantial time derivative, the velocity field and the position vector with respect to the rotating frame; $\Omega \equiv |\boldsymbol{\Omega}|$.

We now choose the origin of the rotating coordinate frame at the centre of the moving sphere. The fluid motion then becomes time-independent and obeys the equation

$$\rho(\boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{U})\cdot\nabla\boldsymbol{v}(\boldsymbol{r}) + 2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r}) = -\nabla p^*(\boldsymbol{r}) + \eta\Delta\boldsymbol{v}(\boldsymbol{r}) \quad \text{for } r > a.$$

Full linearization of this equation with respect to the velocities of both the fluid and the sphere amounts to neglecting the first term on its left-hand side. We shall assume that both velocities are small enough to justify this linearization. The equation of motion now becomes

$$2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r}) = -\nabla p^*(\boldsymbol{r}) + \eta\,\Delta\boldsymbol{v}(\boldsymbol{r}) \quad \text{for } r > a. \tag{2.6}$$

This equation must be supplemented by appropriate boundary conditions at the surface of the sphere. We choose stick boundary conditions:

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{U} + \boldsymbol{\omega}\wedge\boldsymbol{r} \quad \text{for } r = a, \tag{2.7}$$

with $\boldsymbol{\omega}$ the angular velocity of the sphere in the rotating coordinate frame. We consider only the case where the sphere experiences a torque in the direction of the angular velocity $\boldsymbol{\Omega}$. For symmetry reasons the vector $\boldsymbol{\omega}$ must then be parallel to $\boldsymbol{\Omega}$, i.e. $\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\hat{\boldsymbol{\Omega}})\,\hat{\boldsymbol{\Omega}}$ with $\hat{\boldsymbol{\Omega}} \equiv \boldsymbol{\Omega}/\Omega$.

3. Formulation of the problem using induced forces

The concept of induced forces enables us to formulate the problem, posed by (2.2), (2.6) and (2.7), in an alternative way, as follows: we extend the fluid equations within the sphere and write them in the form

$$\left.\begin{aligned} 2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r}) &= -\nabla p^*(\boldsymbol{r}) + \eta\,\Delta\boldsymbol{v}(\boldsymbol{r}) + \boldsymbol{F}_{\text{ind}}(\boldsymbol{r}), & (3.1)\\ \nabla\cdot\boldsymbol{v}(\boldsymbol{r}) &= 0 & (3.2)\end{aligned}\right\} \quad \text{for all } \boldsymbol{r},$$

with $\boldsymbol{F}_{\text{ind}}(\boldsymbol{r}) = 0$ for $r > a$. As extension of the velocity field within the sphere, we choose

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{U} + \boldsymbol{\omega}\wedge\boldsymbol{r} \quad \text{for } r \leqslant a. \tag{3.3}$$

On the reduced hydrostatic pressure we impose the condition

$$p^*(\boldsymbol{r}) = \rho(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})\,\boldsymbol{r}\cdot(\boldsymbol{I} - \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Omega}})\cdot\boldsymbol{r} \quad \text{for } r < a. \tag{3.4}$$

This implies, in view of (2.5), for the extension of the hydrostatic pressure

$$p(\boldsymbol{r}) = \rho\boldsymbol{\Omega}\cdot(\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\Omega})\,\boldsymbol{r}\cdot(\boldsymbol{I} - \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Omega}})\cdot\boldsymbol{r} \quad \text{for } r < a. \tag{3.5}$$

The extensions in (3.4) and (3.5) do not include $r = a$, since the stick boundary condition (2.7) uniquely determines the pressure on the surface of the sphere. In (3.4) and (3.5) $\boldsymbol{I}$ denotes the unit tensor. From substitution of (3.3) and (3.4) in (3.1) it follows that the induced force density $\boldsymbol{F}_{\text{ind}}(\boldsymbol{r})$ is of the form

$$\boldsymbol{F}_{\text{ind}}(\boldsymbol{r}) = a^{-2}\boldsymbol{f}(\hat{\boldsymbol{r}})\,\delta(r-a). \tag{3.6}$$

The factor a^{-2} has been introduced here for convenience, $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$.

If we make use of (2.3), (2.5), (3.1) and (3.3), as well as of Gauss' theorem, we can express the force $\boldsymbol{K}$ exerted by the fluid on the sphere in terms of the induced force density according to

$$\boldsymbol{K} = -\int_{r=a} \mathrm{d}S\,\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r}) = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}) = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}). \tag{3.7}$$

In a similar way we may also relate the torque $\boldsymbol{T}$ that the fluid exerts on the sphere to the induced force density. We have

$$\boldsymbol{T} = -\int_{r=a} \mathrm{d}S\,\boldsymbol{r}\wedge(\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r})) = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\boldsymbol{r}\wedge(\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r})) = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{r}\wedge\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}). \tag{3.8}$$

In order to solve formally the equation of motion for the fluid we introduce Fourier transforms; for example, the velocity field:

$$\boldsymbol{v}(\boldsymbol{k}) \equiv \int \mathrm{d}\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\boldsymbol{v}(\boldsymbol{r}).$$

Equation (3.1) and (3.2) become in wavevector representation

$$(\eta k^2 + 2\rho\boldsymbol{\Omega}\wedge)\,\boldsymbol{v}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}p^*(\boldsymbol{k}) + \boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}), \tag{3.9}$$

$$\boldsymbol{k}\cdot\boldsymbol{v}(\boldsymbol{k}) = 0. \tag{3.10}$$

We now apply the operator $\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$, to both sides of (3.9) and make use of (3.10). We then obtain the equation

$$\eta(k^2 + 2Ta^{-2}(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})]\cdot)\,\boldsymbol{v}(\boldsymbol{k}) = (\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}). \tag{3.11}$$

Here T is the Taylor number defined as

$$T \equiv \frac{\rho\Omega a^2}{\eta}.$$

The tensor $(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})]$ acts on an arbitrary vector $\boldsymbol{s}$ in the following way:

$$\begin{aligned}(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})]\cdot\boldsymbol{s} &= -(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\hat{\boldsymbol{k}}\wedge(\hat{\boldsymbol{k}}\wedge\boldsymbol{s}))]\\ &= (\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[(\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{k}})\,\hat{\boldsymbol{k}}\wedge\boldsymbol{s} - (\hat{\boldsymbol{\Omega}}\cdot(\hat{\boldsymbol{k}}\wedge\boldsymbol{s}))\,\hat{\boldsymbol{k}}]\\ &= (\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{k}})\,\hat{\boldsymbol{k}}\wedge\boldsymbol{s} = -\xi\hat{\boldsymbol{k}}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{s}.\end{aligned} \tag{3.12}$$

Here $\boldsymbol{\epsilon}$ is the Levi-Civita tensor and $\xi \equiv \hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{k}}$. With (3.12) the equation of motion (3.11) can be written as

$$\eta k^2\left(\boldsymbol{I} - \frac{2T\xi}{k^2a^2}\hat{\boldsymbol{k}}\cdot\boldsymbol{\epsilon}\right)\cdot\boldsymbol{v}(\boldsymbol{k}) = (\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}).$$

One easily verifies, using (3.10), that the formal solution of this equation is given by

$$\boldsymbol{v}(\boldsymbol{k}) = \frac{k^2a^4}{\eta(k^4a^4 + 4T^2\xi^2)}\left(\boldsymbol{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} + \frac{2T\xi}{k^2a^2}\hat{\boldsymbol{k}}\cdot\boldsymbol{\epsilon}\right)\cdot\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}). \tag{3.13}$$

This solution implies that the unperturbed fluid is at rest in the rotating frame of reference.

4. Evaluation of the mobility

The aim of our analysis is the evaluation of the translational mobility of the sphere for arbitrary values of the Taylor number. We show in §4.1 that for the problem under consideration translation and rotation of the sphere do not couple. Since the tensors relating $\boldsymbol{U}$ to $\boldsymbol{K}$ and $\boldsymbol{\omega}$ to $\boldsymbol{T}$ have a trivial form, owing to the symmetry of the problem, both the translational mobility μ^{T} and the rotational mobility μ^{R} may be defined as scalar quantities:

$$\boldsymbol{U} \equiv -\mu^{\mathrm{T}} \boldsymbol{K}, \tag{4.1}$$

$$\boldsymbol{\omega} \equiv -\mu^{\mathrm{R}} \boldsymbol{T}. \tag{4.2}$$

For the translational mobility we derive in §§4.2 and 4.3 two different formal expressions. One of these is suitable for numerical evaluation of μ^{T}; the other expression enables us to obtain by an in principle simple calculation the first seven terms in the expansion of μ^{T} in powers of $T^{\frac{1}{2}}$. For the rotational mobility we derive a formal expression in section 4.3 and give the first three terms of its expansion in powers of $T^{\frac{1}{2}}$.

4.1. *A hierarchy of equations for the force multipoles*

As first step in the evaluation of μ^{T} we expand the induced force in irreducible force multipoles and derive for these multipoles a hierarchy of equations.

For the induced force density $\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k})$ we may use the following expansion (see Appendix A)

$$\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}) = \sum_{l=0}^{\infty} (2l+1)!!\,(-i)^l j_l(ka)\,\overline{\hat{\boldsymbol{k}}^l} \odot \boldsymbol{F}^{(l+1)} \tag{4.3}$$

with

$$\boldsymbol{F}^{(l+1)} \equiv (l!)^{-1} \int \mathrm{d}\hat{\boldsymbol{r}}\, \overline{\hat{\boldsymbol{r}}^l} \boldsymbol{f}(\hat{\boldsymbol{r}}). \tag{4.4}$$

$\boldsymbol{F}^{(l+1)}$ is the $(l+1)$th irreducible force multipole moment;

$$(2l+1)!! \equiv (2l+1)(2l-1)\ldots \times 5 \times 3;$$

$j_l(ka)$ is the spherical Bessel function of order l with argument ka. $\overline{\hat{\boldsymbol{k}}^l}$ denotes the irreducible, i.e. symmetric and traceless, tensor of rank l constructed with the vector $\hat{\boldsymbol{k}}$ (see, for example, Hess & Koehler 1980, §1.1). The symbol $\odot$ denotes the full, in this case l-fold, contraction of the tensors $\overline{\hat{\boldsymbol{k}}^l}$ and $\boldsymbol{F}^{(l+1)}$, with the convention that the last index of $\overline{\hat{\boldsymbol{k}}^l}$ is contracted with the first index of $\boldsymbol{F}^{(l+1)}$, etc.

According to (3.6), (3.7) and (4.4), the force $\boldsymbol{K}$ is related to the first force multipole:

$$\boldsymbol{K} = -\boldsymbol{F}^{(1)}. \tag{4.5}$$

Similarly it follows from (3.6), (3.8) and (4.4) that

$$\boldsymbol{T} = a\boldsymbol{\epsilon} : \boldsymbol{F}^{(2)}. \tag{4.6}$$

In order to derive a hierarchy of equations for the force multipoles, we shall determine the so-called velocity surface moments, defined as

$$\overline{\overline{\hat{\boldsymbol{r}}^p}\, \boldsymbol{v}(\boldsymbol{r})}^{S} \equiv (4\pi a^2)^{-1} \int \mathrm{d}\boldsymbol{r}\, \overline{\hat{\boldsymbol{r}}^p}\, \boldsymbol{v}(\boldsymbol{r})\, \delta(r-a), \quad p = 0, 1, 2, \ldots. \tag{4.7}$$

It may be shown that the following identity holds (see Appendix A):

$$\overline{\overline{\hat{\boldsymbol{r}}^p}\, \boldsymbol{v}(\boldsymbol{r})}^{S} = \frac{\mathrm{i}^p}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k}\, j_p(ka)\, \overline{\hat{\boldsymbol{k}}^p}\, \boldsymbol{v}(\boldsymbol{k}), \quad p = 0, 1, 2, \ldots. \tag{4.8}$$

We now substitute (4.3) into (3.13), and the resulting equation into the right-hand side of (4.8). We evaluate the left-hand side of (4.8) with the boundary condition (2.7).

The result of this procedure is the desired hierarchy of equations:

$$U\delta_{p0}+a\boldsymbol{\epsilon}\cdot\boldsymbol{\omega}\delta_{p1} = (4\pi\eta a)^{-1}\sum_{l=0}^{\infty}(1-2\delta_{pl})\,\mathscr{B}^{(p+1,\,l+1)}\odot \boldsymbol{F}^{(l+1)},\quad p=0,1,2,\ldots. \tag{4.9}$$

The factor $1-2\delta_{pl}$ is introduced for convenience. The tensors $\mathscr{B}^{(p+1,\,l+1)}$ of rank $p+l+2$, which will be called *connectors*, are given by

$$\mathscr{B}^{(p+1,\,l+1)} = (\boldsymbol{B}_1^{(p+1,\,l+1)}+\boldsymbol{B}_2^{(p+1,\,l+1)})\,(1-2\delta_{pl}), \tag{4.10}$$

with

$$\boldsymbol{B}_1^{(p+1,\,l+1)} = (2p+1)!!\,(2l+1)!!\,\frac{1}{4\pi}\int \mathrm{d}\hat{\boldsymbol{k}}\,\overline{\hat{\boldsymbol{k}}^p}\,(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\,\overline{\hat{\boldsymbol{k}}^l}\;\mathrm{Re}\,B^{(p+1,\,l+1)}, \tag{4.11}$$

$$\boldsymbol{B}_2^{(p+1,\,l+1)} = (2p+1)!!\,(2l+1)!!\,\frac{1}{4\pi}\int \mathrm{d}\hat{\boldsymbol{k}}\,\overline{\hat{\boldsymbol{k}}^p}\,(\hat{\boldsymbol{k}}\cdot\boldsymbol{\epsilon})\,\overline{\hat{\boldsymbol{k}}^l}\;\mathrm{Im}\,B^{(p+1,\,l+1)}, \tag{4.12}$$

and with

$$B^{(p+1,\,l+1)} = \mathrm{i}^{p-l}\,\frac{2}{\pi}\int_0^{\infty}\mathrm{d}x\,\frac{x^2 j_p(x)\,j_l(x)}{x^4+4T^2\xi^2}\,(x^2+2\mathrm{i}T\xi). \tag{4.13}$$

In (4.11) and (4.12) Re and Im denote the real and imaginary part; in (4.13) $x\equiv ka$.

It is easily checked that $\mathscr{B}^{(p+1,\,l+1)}$ vanishes if $p+l$ is odd, owing to the integration over $\hat{\boldsymbol{k}}$ in (4.11) and (4.12). This allows the separation of hierarchy (4.9) into two sets of equations, namely

$$U\delta_{p0} = (4\pi\eta a)^{-1}\sum_{l=0}^{\infty}(1-2\delta_{pl})\,\mathscr{B}^{(2p+1,\,2l+1)}\odot \boldsymbol{F}^{(2l+1)},\quad p=0,1,2,\ldots, \tag{4.14}$$

and

$$a\boldsymbol{\epsilon}\cdot\boldsymbol{\omega}\delta_{p1} = (4\pi\eta a)^{-1}\sum_{l=1}^{\infty}(1-2\delta_{pl})\,\mathscr{B}^{(2p,\,2l)}\odot \boldsymbol{F}^{(2l)},\quad p=1,2,3,\ldots. \tag{4.15}$$

The above decomposition ensures that translation and rotation do not couple, as is required by symmetry for zero Reynolds number.

From (4.11) and (4.12) it follows that the two connector parts $\boldsymbol{B}_1^{(p+1,\,l+1)}$ and $\boldsymbol{B}_2^{(p+1,\,l+1)}$ satisfy the symmetry relations

$$\boldsymbol{B}_1^{(p+1,\,l+1)} = (-1)^{p+l}\,\widetilde{\boldsymbol{B}_1^{(l+1,\,p+1)}}, \tag{4.16a}$$

$$\boldsymbol{B}_2^{(p+1,\,l+1)} = (-1)^{p+l+1}\,\widetilde{\boldsymbol{B}_2^{(l+1,\,p+1)}}. \tag{4.16b}$$

Here $\tilde{\boldsymbol{C}}$ denotes the generalized transposed of an arbitrary tensor $\boldsymbol{C}$ of rank q, defined as

$$(\tilde{\boldsymbol{C}})_{\alpha_1,\alpha_2,\ldots,\alpha_q} \equiv (\boldsymbol{C})_{\alpha_q,\ldots,\alpha_2,\alpha_1}.$$

We now evaluate the scalar quantity $B^{(p+1,\,l+1)}$. Considering only the case where $p+l$ is even, the integration over $\hat{\boldsymbol{k}}$ in (4.11) and (4.12) may be replaced by twice the integration over those $\hat{\boldsymbol{k}}$ for which $\xi=\boldsymbol{\Omega}\cdot\hat{\boldsymbol{k}}\geqslant 0$. In Appendix B† it is shown how one can obtain by means of complex integration for $B^{(p+1,\,l+1)}$ the expression

$$B^{(p+1,\,l+1)} = \mathrm{i}^{p+1-l}\,z\,j_{\max(p,\,l)}(z)\,h^{(1)}_{\min(p,\,l)}(z), \tag{4.17a}$$

with

$$z = (1+\mathrm{i})\,(T\xi)^{\frac{1}{2}}. \tag{4.17b}$$

In (4.17) $\max(p,l)$ and $\min(p,l)$ denote respectively the larger and smaller integer of the pair p and l. $h_n^{(1)}(z)$ is the first spherical Bessel function of the third kind of order n with argument z (see, for example, Abramowitz & Stegun 1968). For small values of their arguments, $j_n(z)$ and $h_n^{(1)}(z)$ may be expanded in ascending power series in z:

$$j_n(z) = \frac{z^n}{(2n+1)!!} + O(z^{n+2}),$$

$$h_n^{(1)}(z) = -\mathrm{i}\frac{(2n-1)!!}{z^{n+1}} + O(z^{1-n}).$$

One may check by substituting these formulae in (4.17) and the resulting expressions for $B^{(p+1,\,l+1)}$ in (4.11) and (4.12) that the connectors behave for small values of T as

$$\mathscr{B}^{(p+1,\,l+1)} = \mathscr{A}_{p,\,l}\, T^{\frac{1}{2}|p-l|} + O(T^{\frac{1}{2}(|p-l|+1)}). \tag{4.18}$$

4.2. *Systematic evaluation of the translational mobility*

We shall now derive from the set of equations (4.14) an expression for the translational mobility. This expression will enable us to evaluate μ^{T} on the basis of all multipoles up to a certain order, while neglecting the contributions from higher multipoles.

First we shall construct the explicit tensorial form of the irreducible force multipoles and velocity surface moments. By definition the irreducible force multipoles are integrals of the tensor $\overline{\hat{\boldsymbol{r}}^l \boldsymbol{f}(\hat{\boldsymbol{r}})}$ over the unit sphere (cf. (4.4)). According to Hess & Koehler (1980, equation (2.50)), this tensor can for $l \geqslant 1$ be split up into three parts as follows:

$$\overline{\hat{\boldsymbol{r}}^l \boldsymbol{f}(\hat{\boldsymbol{r}})} = \boldsymbol{\Delta}^{(l+1)} \odot \boldsymbol{f}(\hat{\boldsymbol{r}})\,\overline{\hat{\boldsymbol{r}}^l} - \frac{l(2l-1)}{l(2l+1)-1}\,\boldsymbol{\square}^{(l)} \odot (\boldsymbol{f}(\hat{\boldsymbol{r}}) \wedge \overline{\hat{\boldsymbol{r}}^l}) + \frac{2l-1}{2l+1}\,\boldsymbol{\Delta}^{(l)} \odot (\boldsymbol{f}(\hat{\boldsymbol{r}}) \cdot \overline{\hat{\boldsymbol{r}}^l}). \tag{4.19}$$

Here $\boldsymbol{\Delta}^{(l)}$ is a tensor of rank $2l$ that projects out the irreducible part of a tensor of rank l, while $\boldsymbol{\square}^{(l)}$ is defined as

$$(\boldsymbol{\square}^{(l)})_{\mu_1,\ldots,\mu_l,\lambda,\mu_1',\ldots,\mu_l'} \equiv (\boldsymbol{\Delta}^{(l)})_{\mu_1,\ldots,\mu_l,\nu_1,\ldots,\nu_{l-1},\nu_l}(\boldsymbol{\epsilon})_{\nu_l,\lambda,\nu_l'}(\boldsymbol{\Delta}^{(l)})_{\nu_l',\nu_1,\ldots,\nu_{l-1},\mu_1',\ldots,\mu_l'}.$$

The decomposition (4.19) represents a generalization of the standard decomposition of a tensor of rank two into its traceless symmetric and antisymmetric parts and its trace.

Since the connectors are tensors constructed solely with the unit vector $\hat{\boldsymbol{\Omega}}$ (cf. (4.10)–(4.12)), it follows from (4.14) that all force multipoles $\boldsymbol{F}^{(2l+1)}$ are linear in $\hat{\boldsymbol{U}}$. Upon integration over $\hat{\boldsymbol{r}}$, the tensors on the right-hand side of (4.19) therefore each contain one unit vector $\hat{\boldsymbol{U}}$ and the appropriate number of unit vectors $\hat{\boldsymbol{\Omega}}$. Hence we may write

$$\boldsymbol{F}^{(2l+1)} = \sum_{i=1}^{3} F_i^{(2l+1)}\, \boldsymbol{a}_i^{(2l+1)}, \tag{4.20}$$

with

$$\left.\begin{aligned}
\boldsymbol{a}_1^{(2l+1)} &\equiv \left(\frac{(4l+1)!!}{(2l+1)!}\right)^{\frac{1}{2}} \boldsymbol{\Delta}^{(2l+1)} \odot \hat{\boldsymbol{U}}\hat{\boldsymbol{\Omega}}^{2l},\\
\boldsymbol{a}_2^{(2l+1)} &\equiv \left(\frac{(4l-1)!!\,2l}{(2l+1)!}\right)^{\frac{1}{2}} \boldsymbol{\square}^{(2l)} \odot \hat{\boldsymbol{U}}\hat{\boldsymbol{\Omega}}^{2l-1},\\
\boldsymbol{a}_3^{(2l+1)} &\equiv \left(\frac{(4l-1)!!}{(2l-1)!\,(4l+1)}\right)^{\frac{1}{2}} \boldsymbol{\Delta}^{(2l)} \odot \hat{\boldsymbol{U}}\hat{\boldsymbol{\Omega}}^{2l-2}
\end{aligned}\right\}\quad (l \geqslant 1), \tag{4.21a}$$

and for $l = 0$

$$\boldsymbol{a}_1^{(1)} \equiv \hat{\boldsymbol{U}}, \quad \boldsymbol{a}_2^{(1)} \equiv 0, \quad \boldsymbol{a}_3^{(1)} \equiv 0. \tag{4.21b}$$

In Appendix C we show that for all $l \geqslant 1$ the tensors $\boldsymbol{a}_i^{(2l+1)}$ satisfy the relation

$$\widetilde{\boldsymbol{a}_i^{(2l+1)}} \odot \boldsymbol{a}_j^{(2l+1)} \equiv (-1)^{i+1}\,\delta_{ij}, \quad i,j = 1,2,3. \tag{4.22}$$

In view of the identical structure of (4.4) and (4.7), the velocity surface moments can be split up in an analogous way.

We now introduce the scalar quantities $b_{i,j}^{(2p+1,\,2l+1)}$, defined as

$$b_{i,j}^{(2p+1,\,2l+1)} \equiv (1-2\delta_{pl})\,\widetilde{\boldsymbol{a}_i^{(2p+1)}} \odot \mathscr{B}^{(2p+1,\,2l+1)} \odot \boldsymbol{a}_j^{(2l+1)}, \quad i,j = 1,2,3. \tag{4.23}$$

These quantities satisfy the symmetry relation (cf. (4.10), (4.16a,b) and (4.21))

$$b_{i,j}^{(2p+1,\,2l+1)} = b_{j,i}^{(2l+1,\,2p+1)}. \tag{4.24}$$

Using (4.20) and (4.23), we can write (4.14) in the form

$$4\pi\eta a U\delta_{p0} = \sum_{l=0}^{\infty}\sum_{j=1}^{3} b_{i,j}^{(2p+1,\,2l+1)}\,F_j^{(2l+1)}, \quad i = 1,2,3;\ p = 0,1,2,\ldots. \tag{4.25}$$

If we truncate this set of equations at $p = l = M$, we can solve $\boldsymbol{K} = \boldsymbol{F}^{(1)} = -\hat{\boldsymbol{U}}F_1^{(1)}$ from the remaining finite set of equations by application of Cramèr's rule. This yields an expression for the translational mobility which takes into account the influence of the first $M+1$ multipoles with odd superscripts:

$$\mu^{\mathrm{T}}(M) = (4\pi\eta a)^{-1}\begin{cases} b_{1,1}^{(1,1)} & (M=0),\\ |\boldsymbol{b}(M)|\,|\boldsymbol{b}'(M)|^{-1} & (M \geqslant 1),\end{cases} \tag{4.26}$$

where $|\boldsymbol{b}(M)|$ is the determinant of the matrix $\boldsymbol{b}(M)$ with elements $b_{i,j}^{(2\alpha+1,\,2\beta+1)}$, $\alpha,\beta = 0,1,2,\ldots,M$; $i,j = 1,2,3$, and $|\boldsymbol{b}'(M)|$ the determinant of the matrix $\boldsymbol{b}'(M)$ with elements $b_{i,j}^{(2\gamma+1,\,2\delta+1)}$, $\gamma,\delta = 1,2,\ldots,M$; $i,j = 1,2,3$. The true mobility is obtained in the limit $M \to \infty$.

For $M = 0$ the expression for the translational mobility reads explicitly

$$\mu^{\mathrm{T}}(0) = -(4\pi\eta a)^{-1}\,\hat{\boldsymbol{U}}\cdot\mathscr{B}^{(1,1)}\cdot\hat{\boldsymbol{U}}. \tag{4.27}$$

Using (4.10), (4.11) and (4.17), as well as the relations

$$j_0(z) = \frac{\sin z}{z}, \quad h_0^{(1)}(z) = -\frac{\mathrm{i}}{z}\mathrm{e}^{\mathrm{i}z},$$

we may evaluate $\hat{\boldsymbol{U}}\cdot\mathscr{B}^{(1,1)}\cdot\hat{\boldsymbol{U}}$ as follows:

$$\begin{aligned}\hat{\boldsymbol{U}}\cdot\mathscr{B}^{(1,1)}\cdot\hat{\boldsymbol{U}} &= -\int_0^1 \mathrm{d}\xi\,(1-\xi^2)\,\mathrm{Re}\,B^{(1,1)}\\ &= -\tfrac{2}{5}T^{-\frac{1}{2}} - \tfrac{1}{32}T^{-3}\,\mathrm{e}^{-2T^{\frac{1}{2}}}((8T^{\frac{3}{2}}-6T^{\frac{1}{2}}-3)\sin 2T^{\frac{1}{2}} + (8T^{\frac{3}{2}}+12T+6T^{\frac{1}{2}})\cos 2T^{\frac{1}{2}}).\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28), we obtain an expression for μ^{T} that contains the contribution of the first force multipole alone. Its behaviour for large values of T is easily seen to be

$$\mu^{\mathrm{T}}(0) = (10\pi\eta a)^{-1}\,T^{-\frac{1}{2}}.$$

In Appendix E it is shown that $\mu^{\mathrm{T}}(1)$ behaves asymptotically as

$$\mu^{\mathrm{T}}(1) = \tfrac{3}{16}(\eta a T)^{-1}\{1+O(T^{-\frac{1}{2}})\}. \tag{4.29}$$

In the limit $\eta \to 0$ the above expression for $\mu^{\mathrm{T}}(1)$ is equivalent to Stewartson's formula for the ultimate drag experienced by an impulsively started sphere moving in a rotating inviscid fluid (see (5.2)). We have not been able to prove that the same result for the mobility is obtained in the limit $M \to \infty$, i.e. for the true mobility. We note, however, that Stewartson's result has also been derived for the drag on a sphere moving steadily in a rapidly rotating viscous fluid (see, for example, Moore & Saffman 1969), and therefore we feel justified to presume that for T tending to infinity we would indeed find

$$\mu^{\mathrm{T}} = \mu^{\mathrm{T}}(\infty) = \tfrac{3}{16}(\eta a T)^{-1}.$$

Equation (4.29) then implies that the first and third force-multipole moments together determine the asymptotic behaviour of the translational mobility.

In order to estimate the influence of the inclusion of higher multipoles on μ^{T} for arbitrary values of the Taylor number, we have evaluated $\mu^{\mathrm{T}}(M)$ numerically for $M = 0, 1$ and 2. The results of the calculations can be found in §5; they show that the first and third force-multipoles have a dominating influence on the mobility for *all* values of the Taylor number.

4.3. *Power-series expansion for the translational mobility*

In this subsection we shall derive for μ^{T} an alternative expression that is in particular convenient for analysing the behaviour of μ^{T} for small values of T. We shall also give the corresponding expression for μ^{R}.

Using (4.5), we rewrite the set of equations (4.14) as follows:

$$4\pi\eta a \boldsymbol{U} = \boldsymbol{\mathcal{B}}^{(1,1)} \cdot \boldsymbol{K} + \sum_{l=1}^{\infty} \boldsymbol{\mathcal{B}}^{(1,2l+1)} \odot \boldsymbol{\mathcal{F}}^{(2l+1)}, \tag{4.30}$$

$$\boldsymbol{\mathcal{F}}^{(2l+1)} = \boldsymbol{\mathcal{B}}^{(2l+1,2l+1)^{-1}} \odot \left(-\boldsymbol{\mathcal{B}}^{(2l+1,1)} \cdot \boldsymbol{K} + \sum_{\substack{m=1 \\ m \neq l}}^{\infty} \boldsymbol{\mathcal{B}}^{(2l+1,2m+1)} \odot \boldsymbol{\mathcal{F}}^{(2m+1)} \right) \quad (l \geqslant 1). \tag{4.31}$$

Here $\boldsymbol{\mathcal{B}}^{(n,n)^{-1}}$ denotes the inverse of $\boldsymbol{\mathcal{B}}^{(n,n)}$, defined only if $\boldsymbol{\mathcal{B}}^{(n,n)}$ acts on a tensor of rank n that is irreducible in its first $n-1$ indices. By iteration we can eliminate all higher multipoles from the right-hand side of (4.31) in favour of $\boldsymbol{K}$. When the resulting equations are substituted in (4.30) we get an equation of the form (4.1), yielding for μ^{T} the expression

$$\mu^{\mathrm{T}} = (4\pi\eta a)^{-1}\, \overleftrightarrow{\boldsymbol{U}} \cdot \left(-\boldsymbol{\mathcal{B}}^{(1,1)} + \sum_{s=1}^{\infty} \left[\sum_{m_1=1}^{\infty} \cdots \sum_{\substack{m_s=1 \\ m_s \neq m_{s-1}}}^{\infty} \right] \right.$$
$$\left. \times\, \boldsymbol{\mathcal{B}}^{(1,2m_1+1)} \odot \boldsymbol{\mathcal{B}}^{(2m_1+1,2m_1+1)^{-1}} \odot \ldots \odot \boldsymbol{\mathcal{B}}^{(2m_s+1,2m_s+1)^{-1}} \odot \boldsymbol{\mathcal{B}}^{(2m_s+1,1)} \right) \cdot \overleftrightarrow{\boldsymbol{U}}. \tag{4.32}$$

From comparison of (4.26) and (4.32), it follows that for $M \geqslant 2$ the expression for $\mu^{\mathrm{T}}(M)$ given in (4.26) corresponds to a partial resummation of (4.32), involving an infinite number of terms, each of which is a part of the direct contribution of the first $M+1$ force multipoles with odd superscripts to μ^{T}. The expression corresponding to $\mu^{\mathrm{T}}(1)$ is given by

$$\mu^{\mathrm{T}} = (4\pi\eta a)^{-1}\, \overleftrightarrow{\boldsymbol{U}} \cdot \left(-\boldsymbol{\mathcal{B}}^{(1,1)} + \boldsymbol{\mathcal{B}}^{(1,3)} \odot \boldsymbol{\mathcal{B}}^{(3,3)^{-1}} \odot \boldsymbol{\mathcal{B}}^{(3,1)} \right) \cdot \overleftrightarrow{\boldsymbol{U}}. \tag{4.33}$$

With the help of (4.18), we can easily check that the expansion of this expression in powers of $T^{\frac{1}{2}}$ will be correct up to $T^{\frac{7}{2}}$, since contributions from higher multipoles than the quadrupole are at least of order T^4. In Appendix F it is shown that this expansion is given by

$$\mu^{\mathrm{T}} = (6\pi\eta a)^{-1} \left(1 - \tfrac{4}{7} T^{\frac{1}{2}} + \tfrac{8}{45} T^{\frac{3}{2}} - \tfrac{4}{45} T^2 - \tfrac{485504}{2029052025} T^3 + \tfrac{32}{1053} T^{\frac{7}{2}}\right) + O(T^4). \tag{4.34}$$

We note that this series does not contain terms of order T and $T^{\frac{5}{2}}$, and also that the coefficient of the term proportional to T^3 is much smaller than the coefficients of the other terms. Indeed, if we wish to compute values for μ^{T} using (4.36) for comparison with experimental data we may as well neglect this term (cf. table 1 in §5).

We now turn to the other set of equations, the set (4.15), which relates the angular velocity $\boldsymbol{\omega}$ to the force multipoles with even superscripts. We may derive from this set a relation between $\boldsymbol{\omega}$ and $\boldsymbol{T}$ of the form (4.2), yielding for μ^{R} the expression

$$\mu^{\mathrm{R}} = (16\pi\eta a^3)^{-1}\,\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\epsilon}:\Bigg(-\mathscr{B}^{(2,2)}+\sum_{s=1}^{\infty}\Bigg[\sum_{m_1=1}^{\infty}{}' \cdots \sum_{\substack{m_s=1\\ m_s\neq m_{s-1}}}^{\infty}{}'\Bigg] \times \mathscr{B}^{(2,2m_1)}\odot\mathscr{B}^{(2m_1,2m_1)^{-1}}\odot\ldots\odot\mathscr{B}^{(2m_s,2m_s)^{-1}}\odot\mathscr{B}^{(2m_s,2)}\Bigg):\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{\omega}}.$$

Here $\Sigma'^{\infty}_{m_i=1}$ denotes a summation over all integer values $m_i \geqslant 1$ $(i = 1,2,\ldots,s)$ with the proviso that for $m_i = 1$ only the part of $\mathscr{B}^{(\ ,2m_i)}$ or $\mathscr{B}^{(2m_i,\ )}$ symmetric in the corresponding two indices is included in the summation. Expansion of the above expression for μ^{R} up to $T^{\frac{3}{2}}$ yields

$$\mu^{\mathrm{R}} = (8\pi\eta a^3)^{-1}\,(1-\tfrac{8}{45}T^{\frac{3}{2}})+O(T^2).$$

We note that terms proportional to $T^{\frac{1}{2}}$ and T are absent in this expansion.

5. Discussion

When comparing experimental with theoretical results for the drag force experienced by a slowly moving spherical particle, it is convenient to introduce a dimensionless drag, defined as

$$\frac{D}{D_{\mathrm{s}}} \equiv \frac{D}{6\pi\eta a U} = (6\pi\eta a\mu^{\mathrm{T}})^{-1}.$$

The following two expressions for this quantity are available in the literature for the problem under consideration:

Childress (1964) $$\frac{D}{D_{\mathrm{s}}} = 1+\tfrac{4}{7}T^{\frac{1}{2}} \quad \text{for } T \ll 1, \tag{5.1}$$

Stewartson (1952) $$\frac{D}{D_{\mathrm{s}}} = \frac{8}{9\pi}T \quad \text{for } T\to\infty. \tag{5.2}$$

Childress's result is equivalent to the first term in our expansion of μ^{T} in powers of $T^{\frac{1}{2}}$, (4.34); Stewartson's result has already been discussed in §4.2. We shall now compare values for D/D_{s} calculated using the above expressions with experimentally and numerically obtained values for this quantity, as well as with values calculated from (4.26) for $\mu^{\mathrm{T}}(0)$, $\mu^{\mathrm{T}}(1)$ and $\mu^{\mathrm{T}}(2)$.

We first consider Taylor numbers in the range from zero to unity. Column (1) of table 1 contains values for $(D/D_{\mathrm{s}})-1$ calculated with (5.1). Values for this quantity calculated using the first seven terms in the series expansion for μ^{T} are listed in column (2). The results obtained by Dennis *et al.* (1982) from a numerical solution of the full Navier–Stokes equation at $R = 0.12$ are listed in column (3). Column (4) contains values for $(D/D_{\mathrm{s}})-1$ calculated using (4.28). These values are, to within the given accuracy, identical with those obtained from (4.26) for $M = 1$ and $M = 2$. Values for $(D/D_{\mathrm{s}})-1$ obtained by extrapolation of Maxworthy's (1965) experimental data are listed in column 5, together with the errors caused by this extrapolation.

T	(1)	(2)	(3)	(4)	(5)
0.025	0.09	0.10	0.107	0.10	0.09 ± 0.01
0.050	0.13	0.14	—	0.14	0.13 ± 0.02
0.075	0.16	0.18	—	0.18	0.18 ± 0.02
0.10	0.18	0.21	—	0.21	0.22 ± 0.03
0.20	0.26	0.32	—	0.32	0.30 ± 0.04
0.25	0.29	0.37	0.398	0.37	0.37 ± 0.04
0.50	0.40	0.56	0.631	0.57	0.57 ± 0.05
0.75	0.49	0.72	—	0.74	0.75 ± 0.05
1.0	0.57	0.83	—	0.90	—

TABLE 1. Values for $(D/D_s) - 1$, calculated using Childress' result (1), the series expansion (2) and the monopole approximation (4); column 3 giving the results of Dennis *et al.*, and column 5 the experimental values

T	(1)	(2)	(3)
1.0	19.0	19.0	19.0
2.5	10.8	10.8	10.8
5.0	7.47	7.74	7.74
7.5	6.08	6.58	6.58
10.0	5.27	5.94	5.94
25.0	3.33	4.61	4.60
50.0	2.36	4.02	4.02
75.0	1.92	3.78	3.77
100.0	1.67	3.64	3.64
250.0	1.05	3.33	3.32
500.0	7.45×10^{-1}	3.17	3.17
750.0	6.09×10^{-1}	3.11	3.11
1000.0	5.27×10^{-1}	3.07	3.07
10000.0	1.67×10^{-1}	2.90	2.90
100000.0	5.27×10^{-2}	2.85	2.85

TABLE 2. Values for $10D/D_s T$, computed on the basis of the first (1), the first and third (2) and the first, third and fifth (3) force multipole moments

† Appendices B–F containing detailed mathematical arguments are not reproduced here, but interested readers may obtain copies on request from the author or the editor of *JFM*.

It is seen that the results computed using Childress' formula (5.1) are only satisfactory up to $T = 0.2$, whereas those computed using (4.34) are still satisfactory at $T = 0.75$, when compared with the experimental data. The results listed in column (4), which were calculated using the monopole approximation of the induced force, compare very favourably with those in column (5), whereas the numerical results of Dennis *et al.*, column (3), do not agree too well with the experimental values. One should, however, keep in mind that they were calculated for a very small but finite Reynolds number, and must be corrected accordingly.

We have listed in table 2 values for $D/D_s T$ for Taylor numbers above unity. Column 1 contains results computed with the monopole approximation, while in columns (2) and (3) the influences of respectively the third and of the third and fifth force-multipole moments is taken into account. It is seen that values obtained from the monopole approximation deviate by less than 15 % from those obtained from the monopole + quadrupole approximation for $T \leqslant 10.0$. The inclusion of the influence of the hexadecapole moment does not alter significantly the values for $D/D_s T$ obtained from the monopole + quadrupole approximation for *any* value of the Taylor number. As anticipated, Stewartson's limiting value $8/9\pi$ for $D/D_s T$ is approached more and more closely for higher and higher values of T in columns 2 and 3.

Maxworthy (1970) has determined experimentally that D/D_s behaves for large values of T as

$$\frac{D}{D_s} = (0.43 \pm 0.01) T^{1.00 \pm 0.01}.$$

In an attempt to explain the discrepancy between this result and Stewartson's expression (5.2), Hocking, Moore & Walton (1979) have analysed the influence of the finite axial size of the container, used by Maxworthy for his experiments, on the value of the drag. The results of their analysis suggest this is not the main cause of the discrepancy. To determine whether (and, if so, to what extent) nonlinear effects, in particular the influence of momentum convection, are responsible for the discrepancy, it would be interesting to compare the values of $D/D_s T$ for Taylor numbers above unity, listed in column (3) of table 2, with experimentally obtained values for this quantity. Except for the asymptotic value, quoted above, such experimental results have not been found in the presently available literature.

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Appendix A. Irreducible force multipoles and velocity surface moments

In this appendix we shall derive for the induced force density $\boldsymbol{F}_{\text{ind}}(\boldsymbol{k})$ the expansion (4.3), and for the velocity surface moments the expression (4.8). We shall make use of the following identity:

$$\frac{\overline{\mathrm{d}^l}}{\mathrm{d}\boldsymbol{k}^l} f(k) = \overline{\hat{\boldsymbol{k}}^l}\, k^l \left(\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\right)^l f(k), \quad l \in \mathbb{N}$$

with $f(k)$ an arbitrary function of $k = |\boldsymbol{k}|$. For the case where $f(k) = j_0(ka)$, this identity becomes, using also Rayleigh's formula (see Abramowitz & Stegun 1968),

$$\frac{\overline{\mathrm{d}^l}}{\mathrm{d}\boldsymbol{k}^l} j_0(ka) = (-a)^l\, \overline{\hat{\boldsymbol{k}}^l}\, j_l(ka). \tag{A 1}$$

We shall also use the identity

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!}\, \overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l}, \tag{A 2}$$

which may be derived by combination of the expansion

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}')$$

with $P_l(x)$ the Legendre polynomial of degree l (see, for example, Jackson 1975, equations (3.62) and (3.117)), and the relation

$$P_l(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}') = \frac{(2l-1)!!}{l!}\, \overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l}$$

(see Hess & Koehler 1980, equation (4.21)).

We obtain from (3.6), (4.4), (A 1) and (A 2):

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}) &= \int \mathrm{d}\hat{\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{\boldsymbol{r}}} \boldsymbol{f}(\hat{\boldsymbol{r}}) \\
&= \int \mathrm{d}\hat{\boldsymbol{r}} \int \mathrm{d}\hat{\boldsymbol{r}}'\, \delta(\hat{\boldsymbol{r}}-\hat{\boldsymbol{r}}')\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{\boldsymbol{r}}'} \boldsymbol{f}(\hat{\boldsymbol{r}}') \\
&= \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!} \frac{1}{4\pi} \int \mathrm{d}\hat{\boldsymbol{r}}'\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{\boldsymbol{r}}'}\, \overline{\hat{\boldsymbol{r}}'^{l}} \odot \int \mathrm{d}\hat{\boldsymbol{r}}\, \overline{\hat{\boldsymbol{r}}^{l}} \boldsymbol{f}(\hat{\boldsymbol{r}}) \\
&= \sum_{l=0}^{\infty} (2l+1)!! \left(\frac{\mathrm{i}}{a}\right)^{l} \left[\overline{\frac{\partial^{l}}{\partial \boldsymbol{k}^{l}}} \frac{1}{4\pi} \int \mathrm{d}\hat{\boldsymbol{r}}'\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{\boldsymbol{r}}'}\right] \odot \boldsymbol{F}^{(l+1)} \\
&= \sum_{l=0}^{\infty} (2l+1)!!\, (-\mathrm{i})^{l} j_{l}(ka)\, \overline{\hat{\boldsymbol{k}}^{l}} \odot \boldsymbol{F}^{(l+1)},
\end{aligned}$$

which is the desired expansion. Combining (4.7) and (A 1), we may derive (4.8) in the following way:

$$\begin{aligned}
\overline{\overline{\hat{\boldsymbol{r}}^{p}}\, \boldsymbol{v}(\boldsymbol{r})}^{S} &= \frac{1}{4\pi a^{2}} \int \mathrm{d}\boldsymbol{r}\, \overline{\hat{\boldsymbol{r}}^{p}}\, \delta(r-a) \frac{1}{(2\pi)^{3}} \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \boldsymbol{v}(\boldsymbol{k}) \\
&= \frac{1}{(2\pi)^{3}} \int \mathrm{d}\boldsymbol{k} \left[\frac{1}{4\pi} \int \mathrm{d}\hat{\boldsymbol{r}}\, \mathrm{e}^{\mathrm{i}a\boldsymbol{k}\cdot\hat{\boldsymbol{r}}}\, \overline{\hat{\boldsymbol{r}}^{p}}\right] \boldsymbol{v}(\boldsymbol{k}) \\
&= \frac{\mathrm{i}^{p}}{(2\pi)^{3}} \int \mathrm{d}\boldsymbol{k}\, j_{p}(ka)\, \overline{\hat{\boldsymbol{k}}^{p}}\, \boldsymbol{v}(\boldsymbol{k}).
\end{aligned}$$

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